

THE KATO-NAKAYAMA SPACE AS A TRANSCENDENTAL ROOT STACK

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ABSTRACT. We give a functorial description of the Kato-Nakayama space of a fine saturated log analytic space that is similar in spirit to the functorial description of root stacks. As a consequence we get a global description of the comparison map constructed in [CSST] from the Kato-Nakayama space to the (topological) infinite root stack.

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1. INTRODUCTION

Let X be a fine saturated log scheme, locally of finite type over \mathbb{C} , or a log analytic space. There have been a few constructions aimed at capturing the “log geometry” of X in more familiar forms. Two of those are the “Kato-Nakayama” space X_{\log} (a topological space, introduced in [KN99]), and the “infinite root stack” $\varprojlim X$ (a pro-algebraic stack, introduced in [TV]). As mentioned in the introduction of [TV], the latter is, morally, an “algebraic version” of the former.

Building on this idea, in the paper [CSST] by Carchedi, Scherotzke, Sibilla and the first author it is shown that there is a canonical morphism $\Phi_X: X_{\log} \rightarrow \varprojlim X_{\text{top}}$ from the Kato-Nakayama space to the topological realization of the infinite root stack of X , that is moreover a “profinite equivalence”. In that paper the morphism is constructed locally on X , in presence of a Kato chart for the log structure, and then globalized by gluing [CSST, Section 4].

Two natural questions arise from this construction.

- (1) Is there a global definition of the morphism Φ_X ? For example, one could hope to use a “functor of points” point of view, describing the objects of the groupoid $\varprojlim X_{\text{top}}$, and then producing an object of $\varprojlim X_{\text{top}}(X_{\log})$.
- (2) Over X_{\log} there is a sheaf of rings \mathcal{O}_X^{\log} that makes the projection $X_{\log} \rightarrow X$ into a map of ringed spaces, and over $\varprojlim X_{\text{top}}$ there is a

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natural structure sheaf \mathcal{O}_∞ . Does Φ_X extend to a morphism of ringed topological stacks?

In this paper we answer the first question, and give such a global construction of Φ_X . In future work we plan to investigate question (2) and “coherent sheaves” on X_{\log} , and to connect these with parabolic sheaves with real weights on X (parabolic sheaves with rational weights on X correspond to quasi-coherent sheaves on $\sqrt[n]{X}$).

In order to answer question (1) we will give a functorial description of $\sqrt[n]{X}_{\text{top}}$, and produce an object of the corresponding kind on the topological space X_{\log} . We will do so by giving a “root stack” functorial definition of X_{\log} , that is closely related to the functorial description given in [IKN05], from which it will be apparent that an object parametrized by the Kato-Nakayama space induces compatible n -th roots for each positive integer n .

In some more detail: we use the point of view of [BV12], according to which a log structure on X can be seen as a symmetric monoidal functor $L: A \rightarrow [\mathbb{C}/\mathbb{C}^\times]_X$ from a sheaf of monoids to a stack of line bundles with global section on open subsets of X . Root stacks parametrize liftings of the functor L along n -th power maps $\wedge n: [\mathbb{C}/\mathbb{C}^\times] \rightarrow [\mathbb{C}/\mathbb{C}^\times]$, induced by $z \mapsto z^n$ on both the space \mathbb{C} and the group \mathbb{C}^\times (and corresponding to raising both the line bundle and the global section to the n -th power). The Kato-Nakayama space turns out to parametrize similar liftings, in which instead of extracting n -th roots for a fixed n we are in some sense extracting a “logarithm”, i.e. we are lifting the log structure along a sort of “exponential” $\mathcal{H} \rightarrow [\mathbb{C}/\mathbb{C}^\times]$, where \mathcal{H} is a certain stack that we will describe. Since there is a factorization

$$\mathcal{H} \rightarrow [\mathbb{C}/\mathbb{C}^\times] \xrightarrow{\wedge n} [\mathbb{C}/\mathbb{C}^\times]$$

for every n , every such lifting will in particular give a compatible system of roots.

In order to describe the functor of points of $\sqrt[n]{X}_{\text{top}}$ we consider some sort of complex-valued log structures on topological spaces. Since much of the formalism turns out to be quite general, we give the definitions and prove some basic facts by using the language of topoi.

We hope that some of the formalism developed here might be useful in the future, in the answer to question (2) or other related matters.

Outline. Section 2 is about log structures on what we call “monoided topoi”; we adapt the general formalism of [GMa, GMb], extending it to include the case of the étale topology on schemes (in the two papers just mentioned the “spaces” are actual topological spaces). We give definitions in analogy to those of Kato in the algebraic case [Kat89] and relate it to the alternative “Deligne–Faltings” language introduced in [BV12]. We describe spaces and stacks of charts, and consider root stacks in this general framework. At the end of the section we specialize our setting to schemes locally of finite type over \mathbb{C} with the étale topology, analytic spaces and topological spaces.

In Section 3 we describe a functorial interpretation of the Kato-Nakayama space, that is the translation in the Deligne-Faltings language of the one given in [IKN05, Section 1] (recalled in this paper as Theorem 3.3). We describe the natural “charts” for X_{\log} that correspond to this description,

and we produce a globally defined morphism from the Kato-Nakayama space to the topological infinite root stack.

To conclude, in Section 4 we point out that the Kato-Nakayama construction can be extended to algebraic (and analytic) stacks, and we check that the morphism to the infinite root stack produced in the previous section coincides with the one of [CSST].

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Notations and conventions. We assume some familiarity with log geometry. For an introduction, see for example [Kat89] or [CSST, Appendix]. We are mostly interested in fine and saturated log structures.

By the results of [BV12], for schemes the “Kato language” is equivalent to the “Deligne-Faltings” language.

All our monoids will be commutative. If P is a monoid and X is a monoid with some additional structure (for example a topological space), we will denote by $X(P)$ the object $\mathrm{Hom}_{\mathrm{Mon}}(P, X)$ with its naturally induced additional structure. For example we can take $X = \mathbb{R}_{\geq 0}$ to be the topological monoid of non-negative real numbers with respect to multiplication, and then $\mathbb{R}_{\geq 0}(P)$ will denote the topological monoid $\mathrm{Hom}(P, \mathbb{R}_{\geq 0})$. We will denote by \widehat{P} the diagonalizable group scheme $\mathrm{Spec} k[P^{\mathrm{gp}}]$ associated with the abelian group P^{gp} . The sheafification of the constant presheaf with sections P will be denoted by \underline{P} .

For symmetric monoidal categories we adopt the language and conventions of [BV12] (see in particular Section 2.4).

All our algebraic spaces will be locally separated. If X is a scheme (or algebraic space) over \mathbb{C} , we write $X_{\mathrm{\acute{e}t}}$ for the small étale site of X . If X is an analytic space we will denote by \mathcal{A}_X the small analytic (or classical) site of X , and if T is a topological space we will denote by \mathcal{C}_T the classical site of T . If X is a locally separated algebraic space that is locally of finite type over \mathbb{C} , we will denote by $(X_{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ (sometimes just by X_{an} by abuse of notation) its analytification as an analytic space, and by X_{top} the underlying topological space of X_{an} . Although X_{top} and X_{an} are the *same* topological space, we prefer to keep the two symbols distinct, so that it will be clear whether we are in the analytic or topological world.

We will denote by \mathcal{O}_X the structure sheaf of either a scheme (or algebraic space) or of an analytic space.

2. LOG STRUCTURES ON MONOIDED TOPOI

In order to give a functorial interpretation of the topological infinite root stack $\sqrt[n]{X}_{\mathrm{top}} := \varprojlim_n \sqrt[n]{X}_{\mathrm{top}}$ (whose definition is recalled later) of a fine saturated log analytic space X , we need to introduce a notion of log structures on a topological space. Since X could be of the form Y_{an} for a fine saturated log algebraic space Y locally of finite type over \mathbb{C} , we also take the intermediate step of discussing log structures on analytic spaces in the language of [BV12].

Since the basic definitions and facts can be formulated in the language of topoi with a sheaf of monoids, we choose to discuss the basics in this generality.

The proofs in this section will be somewhat terse. The interested reader can look at the more detailed treatment of [BV12], in the algebraic case, or the one of [GMa, GMb] in an axiomatic framework of “log spaces”.

If \mathfrak{X} is a topos, we denote by \mathbf{pt} its terminal object. A *covering* of \mathfrak{X} is a set $\{U_i\}$ of objects of \mathfrak{X} such the morphism $\bigsqcup_i U_i \rightarrow \mathbf{pt}$ is an epimorphism.

If U is an object of \mathfrak{X} , we denote by (\mathfrak{X}/U) the comma category, which is well known to be a topos. If A is a monoid in \mathfrak{X} , we will denote by $A(U)$ the monoid of arrows $U \rightarrow A$.

Definition 2.1. A *monoided topos* is a pair $(\mathfrak{X}, \mathcal{O})$ consisting of a topos \mathfrak{X} and a monoid \mathcal{O} in \mathfrak{X} .

We denote by \mathcal{O}^\times the subobject of \mathcal{O} consisting of invertible sections.

Remark 2.2. The strange term “monoided” is a back-formation inspired from “ringed”. We will reserve “monoidal” for symmetric monoidal categories.

Examples 2.3. We will later specialize this formalism to three concrete situations, that the reader might want to have in mind right away.

If X is a scheme (or algebraic space) over \mathbb{C} , we will consider the topos of sheaves on the small étale site of X , with its structure sheaf.

If X is an analytic space, we will consider the topos of sheaves on X for the analytic topology, and its sheaf of analytic functions.

Finally, if X is a topological space, we will consider the topos of sheaves on X and the sheaf of continuous complex valued functions.

All these sheaves of commutative rings are seen as sheaves of monoids via multiplication.

2.1. Kato and DF log structures.

Definition 2.4. A *Kato pre-log structure* on a monoided topos $(\mathfrak{X}, \mathcal{O})$ is a pair (M, α) consisting of a monoid M in \mathfrak{X} and a homomorphism $\alpha: M \rightarrow \mathcal{O}$.

A Kato pre-log structure is called a *Kato log structure* if the restriction $\alpha|_{\alpha^{-1}\mathcal{O}^\times}: \alpha^{-1}\mathcal{O}^\times \rightarrow \mathcal{O}^\times$ is an isomorphism.

A general definition of log structures in the sense of Kato in a “category of spaces” appears in [GMa], although the formalism considered in that article does not allow for Grothendieck topologies that are not “classical” (for example the étale topology on a scheme).

There is a category of Kato log structures on a monoided topos $(\mathfrak{X}, \mathcal{O})$. Maps $(M, \alpha) \rightarrow (N, \beta)$ are homomorphisms of monoids $\phi: M \rightarrow N$ such that the composite $\beta \circ \phi$ coincides with α .

As usual, one can obtain a Kato log structure from a Kato pre-log structure by taking the amalgamated sum $M \oplus_{\alpha^{-1}\mathcal{O}^\times} \mathcal{O}^\times$ and the induced map to \mathcal{O} . A Kato log structure (M, α) on $(\mathfrak{X}, \mathcal{O})$ will be called *quasi-integral* if the action of \mathcal{O}^\times on M is free (i.e. does not have any non-trivial stabilizers). The quotient sheaf $\overline{M} = M/\mathcal{O}^\times$ is usually called the *characteristic monoid* of the log structure.

Let $\mathrm{Div}_{\mathfrak{X}}$ denote the quotient stack $[\mathcal{O}/\mathcal{O}^\times]$ over the site $(\mathfrak{X}, \mathcal{O})$, where the group \mathcal{O}^\times acts on \mathcal{O} via the inclusion. This is a symmetric monoidal stack, with monoidal structure induced by the multiplication of \mathcal{O} . The notation is taken from [BV12] (see in particular Examples 2.8 and Remark 2.9), and reflects the fact that in the algebraic case $[\mathcal{O}/\mathcal{O}^\times]$ is the stack of “generalized Cartier divisors” (i.e. line bundles with a global section).

Definition 2.5. A *Deligne-Faltings* (DF for short) *log structure* on a monoidal topos $(\mathfrak{X}, \mathcal{O})$ is a pair (A, L) where A is a sharp monoid in \mathfrak{X} and $L: A \rightarrow \mathrm{Div}_{\mathfrak{X}}$ is a symmetric monoidal functor with trivial kernel.

In the definition above, “with trivial kernel” means that if a section $a \in A(U)$ is such that $L(a)$ is invertible in $\mathrm{Div}_{\mathfrak{X}}$ (in the monoidal sense), then $a = 0$. This definition is a particular case of that of a “Deligne-Faltings object” of [BV12].

There is a category of DF log structures on a monoidal topos $(\mathfrak{X}, \mathcal{O})$. Maps $(A, L) \rightarrow (B, K)$ are given by homomorphisms of monoids $\phi: A \rightarrow B$ with a natural equivalence $\Phi: K \circ \phi \cong L$.

Proposition 2.6. *Let $(\mathfrak{X}, \mathcal{O})$ be a monoidal topos. There is an equivalence of categories between quasi-integral Kato log structures and DF log structures on $(\mathfrak{X}, \mathcal{O})$.*

Proof. Given a quasi-integral Kato log structure (M, α) , we produce a DF log structure by dividing $\alpha: M \rightarrow \mathcal{O}$ by the action of \mathcal{O}^\times in the stacky sense. We obtain a morphism of symmetric monoidal stacks $\bar{\alpha}: \bar{M} = [M/\mathcal{O}^\times] \rightarrow [\mathcal{O}/\mathcal{O}^\times] = \mathrm{Div}_{\mathfrak{X}}$, and by assumption \bar{M} is equivalent to a sheaf. The pair $(\bar{M}, \bar{\alpha})$ is a DF log structure.

Conversely, starting from a DF log structure (A, L) we consider the fibered product $M = A \times_{\mathrm{Div}_{\mathfrak{X}}} \mathcal{O}$ and the induced symmetric monoidal morphism $\alpha: M \rightarrow \mathcal{O}$. Since $\mathcal{O} \rightarrow \mathrm{Div}_{\mathfrak{X}}$ is a \mathcal{O}^\times -torsor, the stack M is equivalent to a sheaf (of monoids), and the pair (M, α) is a quasi-integral Kato log structure.

We leave it to the reader to check that these two constructions are quasi-inverses (see also [BV12, Theorem 3.6]). \square

For the rest of the paper all Kato log structures will be quasi-integral, we will drop the “Kato” and “DF”, and just talk about log structures, and we will switch freely between the two notions and notations.

2.2. Coherent sheaves of monoids and charts. Let A be a monoid in a topos \mathfrak{X} , P a finitely generated monoid and $P \rightarrow A(\mathfrak{X})$ a homomorphism of monoids. If \underline{P} is the locally constant sheaf given by P there is an induced homomorphism of sheaves of monoids $\phi: \underline{P} \rightarrow A$.

Definition 2.7.

- (1) A homomorphism $P \rightarrow A(\mathfrak{X})$ is a *chart* if the induced homomorphism $\phi: \underline{P} \rightarrow A$ is a cokernel, or, equivalently, if the induced homomorphism $\underline{P}/\ker \phi \rightarrow A$ is an isomorphism.
- (2) An *atlas* for A is a covering $\{U_i\}$ of \mathfrak{X} together with a chart $P_i \rightarrow A(U_i)$ for the pullback of A to (\mathfrak{X}/U_i) for each i .
- (3) The object A is called *coherent* if it admits an atlas.

Definition 2.8. A monoid A in \mathfrak{X} is called *sharp*, or *integral*, or *torsion-free*, or *saturated*, if for each object U of \mathfrak{X} the monoid $A(U)$ has the homonymous property.

As usual, “fine” will mean “integral and finitely generated” (and also coherent, when applied to a monoid A in \mathfrak{X}).

It is easy to see that if \mathfrak{X} has enough points, then A has any of the properties above if and only if for each point p of \mathfrak{X} , the stalk A_p has the homonymous property.

Proposition 2.9. *Let A be a sharp coherent monoid in \mathfrak{X} . Assume that A is fine, or fine and torsion-free, or fine and saturated. Then A has an atlas $\{P_i \rightarrow A(U_i)\}$ in which every P_i has the homonymous property.*

If \mathfrak{X} has enough points then one can give a proof along the lines of [BV12, Proposition 3.15].

Proof. Assume that $\phi: P \rightarrow A(U)$ is a chart for A on U , and set $Q = A(U)$. We will show that $\psi: Q \rightarrow A(U)$ (the identity) is also a chart for A .

We have a diagram $\underline{P}/\ker \phi \rightarrow \underline{Q}/\ker \psi \rightarrow A$, and the composite is an isomorphism by assumption, so $\underline{Q}/\ker \psi \rightarrow A$ is surjective. Let us check that it is also injective.

For an object V (on which we will localize further several times), two sections x and y of $(\underline{Q}/\ker \psi)(V)$ that map to the same section of $A(V)$ are locally represented by two elements a, b of Q , that map to the same element in $A(V)$. Since $\underline{P} \rightarrow A$ is surjective, locally the sections $a, b \in Q = A(U)$ are images of $p, p' \in P$. Since $\underline{P}/\ker \phi \rightarrow A$ is a cokernel and p and p' map to the same element in $A(V)$, there locally exist sections $h, k \in (\ker \phi)(V)$ such that $p + h = p' + k$ in $\underline{P}(V)$. If h' and k' are the images of h and k in $\underline{Q}(V)$ (and note that they are sections of $\ker \psi$), we have $a|_V + h' = b|_V + k'$. This implies that the two sections $x, y \in (\underline{Q}/\ker \psi)(V)$ considered at the beginning are locally equal, and shows that $\underline{Q}/\ker \psi \rightarrow A$ is injective. \square

Now let $(\mathfrak{X}, \mathcal{O})$ be a monoidal topos.

Definition 2.10. A DF log structure (A, L) on $(\mathfrak{X}, \mathcal{O})$ is *coherent*, or *fine*, or *torsion-free*, or *saturated*, if A has the homonymous property.

Notice that if (A, L) is a DF log structure in $(\mathfrak{X}, \mathcal{O})$, then A is automatically sharp. Hence, since every sharp fine saturated monoid is torsion-free, a fine saturated DF log structure is automatically torsion-free.

Given a symmetric monoidal functor $h: P \rightarrow \text{Div}_{\mathfrak{X}}(\mathfrak{X})$, by sheafifying and killing the kernel (in the monoidal sense) of the resulting functor we obtain a DF log structure $(P, h)^a$ (see Propositions 2.4 and 2.10 of [BV12]). If (A, L) is a DF log structure on $(\mathfrak{X}, \mathcal{O})$, $k: P \rightarrow A(\mathfrak{X})$ is a homomorphism of monoids, and $h = L \circ k: P \rightarrow \text{Div}_{\mathfrak{X}}(\mathfrak{X})$ is the composite, we obtain a morphism of DF log structures $(P, h)^a \rightarrow (A, L)$.

Proposition 2.11. *The homomorphism $h: P \rightarrow A(\mathfrak{X})$ is a chart if and only if the induced morphism $(P, h)^a \rightarrow (A, L)$ is an isomorphism.*

Proof. Let $K \subseteq \underline{P}$ be the kernel of h , which equals the kernel of the symmetric monoidal functor $\underline{P} \rightarrow \text{Div}_{\mathfrak{X}}$ induced by h ; by construction, the sheaf

of monoids of the DF log structure is \underline{P}/K . Hence, h is a cokernel if and only if the morphism of DF log structures $(P, h)^a \rightarrow (A, L)$ induces an isomorphism on sheaves of monoids. On the other hand since $\mathrm{Div}_{\mathfrak{X}}$ is fibered in groupoids we have that a morphism of DF log structures $(B, M) \rightarrow (A, L)$ is an isomorphism if and only if the corresponding homomorphism $B \rightarrow A$ is an isomorphism, and this concludes the proof. \square

In our general context Kato charts also work very well.

Definition 2.12. Let (M, α) be a Kato log structure on $(\mathfrak{X}, \mathcal{O})$.

- (1) A *Kato chart* on (M, α) is a homomorphism of monoids $P \rightarrow M(\mathfrak{X})$ such that composite $P \rightarrow M(\mathfrak{X}) \rightarrow \overline{M}(\mathfrak{X})$ is a DF chart.
- (2) A *Kato atlas* for (M, α) is a covering $\{U_i\}$ of \mathfrak{X} together with a chart $P_i \rightarrow M(U_i)$ for the pullback of (M, α) to (\mathfrak{X}/U_i) for each i .

One can give examples of fine log structures that do not admit Kato atlases; however, we have the following.

Proposition 2.13. *Let P be a fine torsion-free monoid. Any symmetric monoidal functor $P \rightarrow \mathrm{Div}_{\mathfrak{X}}(\mathfrak{X})$ lifts, locally on \mathfrak{X} , to a homomorphism $P \rightarrow \mathcal{O}(\mathfrak{X})$.*

Proof. Consider the induced homomorphism $\underline{P} \rightarrow \mathrm{Div}_{\mathfrak{X}}$, and call the E the pullback of \mathcal{O} to \underline{P} , which is an integral sheaf of monoids. Then \mathcal{O}^\times is contained in E , acts freely on E , and $E/\mathcal{O}^\times = \underline{P}$. Passing to the associated sheaves of groups is a colimit-preserving functor, being a left adjoint. Furthermore, since E is integral we have that the composite $\mathcal{O}^\times \rightarrow E \rightarrow E^{\mathrm{gp}}$ is injective; so we have an exact sequence

$$0 \rightarrow \mathcal{O}^\times \rightarrow E^{\mathrm{gp}} \rightarrow \underline{P}^{\mathrm{gp}} \rightarrow 0.$$

But $\underline{P}^{\mathrm{gp}}$ is a free abelian group object in \mathfrak{X} , because P is fine and torsion-free; hence by passing to a covering of \mathfrak{X} we may assume that there is a splitting homomorphism $\underline{P}^{\mathrm{gp}} \rightarrow E^{\mathrm{gp}}$. Since E is an \mathcal{O}^\times -torsor over \underline{P} , and E^{gp} is an \mathcal{O}^\times -torsor over $\underline{P}^{\mathrm{gp}}$, we have that E is the inverse image of $\underline{P} \subseteq \underline{P}^{\mathrm{gp}}$ inside E^{gp} . It follows that the splitting $\underline{P}^{\mathrm{gp}} \rightarrow E^{\mathrm{gp}}$ restricts to a splitting $\underline{P} \rightarrow E$; the composite $\underline{P} \rightarrow E \rightarrow M$ gives a homomorphism $P \rightarrow M(\mathfrak{X})$, which is the desired Kato chart. \square

Corollary 2.14. *A fine torsion-free log structure has a Kato atlas.*

Proof. Let (M, α) be a Kato log structure on $(\mathfrak{X}, \mathcal{O})$; call (\overline{M}, L) the corresponding DF log structure. We may assume that (\overline{M}, L) has a DF chart $P \rightarrow \overline{M}(\mathfrak{X})$; by Proposition 2.9 we may assume that P is integral and torsion-free. Then the result follows from Proposition 2.13. \square

2.3. Functoriality. Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ be monoided topoi.

Definition 2.15. A morphism of monoided topoi $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ is a morphism of topoi $F: \mathfrak{X} \rightarrow \mathfrak{Y}$ together with a homomorphism of sheaves of monoids $F^{-1}\mathcal{O}_{\mathfrak{Y}} \rightarrow \mathcal{O}_{\mathfrak{X}}$.

Given a morphism $f: (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ and a log structure (M, α) on $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$, there is a log structure $f^{-1}(M, \alpha)$ obtained by pullback: the morphism $\alpha: M \rightarrow \mathcal{O}_{\mathfrak{Y}}$ induces a pre-log structure $f^{-1}\alpha: f^{-1}M \rightarrow f^{-1}\mathcal{O}_{\mathfrak{Y}} \rightarrow \mathcal{O}_{\mathfrak{X}}$, and $f^{-1}(M, \alpha)$ is the associated log structure.

Remark 2.16. If one uses DF log structures, the pullback is simply defined by the composite $f^{-1}A \rightarrow f^{-1}\mathrm{Div}_{\mathcal{Y}} \rightarrow \mathrm{Div}_{\mathcal{X}}$ (see [BV12, Section 3.2] for details—the arguments generalize without any difficulty), and there is no need to change the sheaf of monoids.

If $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is equipped with a log structure (M, α) and $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ with (N, β) , a morphism between the “log monoided topoi” $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}, M, \alpha) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}, N, \beta)$ is a morphism of monoided topoi $f: (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ together with a morphism of log structures $f^{-1}(N, \beta) \rightarrow (M, \alpha)$.

Such a morphism of log monoided topoi is said to be *strict* if $f^{-1}(N, \beta) \rightarrow (M, \alpha)$ is an isomorphism.

2.4. Spaces of Kato charts and stacks of DF charts.

Assumption 2.17. We fix a category **Esp** admitting finite inverse limits, equipped with a subcanonical Grothendieck topology, a monoid object \mathbb{A}^1 in **Esp**, and a class of arrows in **Esp**, called *small*.

We assume the following conditions.

- (1) An isomorphism is small.
- (2) The class of small arrows is closed under composition.
- (3) The class of small arrows is closed under pullback, that is, if $X \rightarrow Y$ and $Y' \rightarrow Y$ are arrows and $X \rightarrow Y$ is small, the projection $Y' \times_Y X \rightarrow Y'$ is also small.
- (4) If U is an object of **Esp**, any covering of U has a refinement $\{U_i \rightarrow U\}$ in which every map $U_i \rightarrow U$ is small.

We think about the objects of **Esp** as some sort of “spaces” (as in [GMa, GMb]—the notation is borrowed from there).

For the purposes of our paper, we are interested in the following three classes of examples.

- (1) **Esp** is the category of schemes over a fixed base field k . The topology is the étale topology, $\mathbb{A}^1 = \mathbb{A}_k^1$, and small arrows are étale maps.
- (2) **Esp** is the category of analytic spaces, the topology is the classical topology, $\mathbb{A}^1 = \mathbb{C}$, and small arrows are open embeddings.
- (3) **Esp** is the category of topological spaces, the topology is the classical topology, $\mathbb{A}^1 = \mathbb{C}$, and small arrows are open embeddings.

Denote by \mathcal{O} the sheaf of morphisms towards \mathbb{A}^1 . If $\mathcal{O}^\times \subseteq \mathcal{O}$ is the subsheaf of invertible sections, then \mathcal{O}^\times is represented by a subobject $\mathbb{G}_m \subseteq \mathbb{A}^1$: if **pt** is the terminal object of **Esp** and **pt** $\rightarrow \mathbb{A}^1$ is the morphism corresponding to the identity in \mathbb{A}^1 , then \mathbb{G}_m is represented by the fibered product $(\mathbb{A}^1 \times \mathbb{A}^1) \times_{\mathbb{A}^1} \mathbf{pt}$, where the morphism $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the operation in \mathbb{A}^1 .

Given an object $S \in \mathbf{Esp}$, we can consider the induced small site S_{sm} of S : its objects are small arrows $U \rightarrow S$, with the obvious morphisms, and the topology is induced by the topology of **Esp**. We denote by \mathfrak{X}_S the topos of sheaves on S_{sm} , together with the sheaf \mathcal{O}_S of morphisms to the monoid object \mathbb{A}^1 . This gives a monoided topos $(\mathfrak{X}_S, \mathcal{O}_S)$, and we can consider log structures on it, morphisms between these objects, and so on.

If $f: S \rightarrow T$ is an arrow in **Esp**, fibered product induces a functor $T_{\mathrm{sm}} \rightarrow S_{\mathrm{sm}}$, which in turn gives a morphism of topoi $(f_*, f^{-1}): \mathfrak{X}_S \rightarrow \mathfrak{X}_T$.

A *log space* (S, M, α) in **Esp** will be an object $S \in \mathbf{Esp}$, together with a log structure on the monoided topos \mathfrak{X}_S as above (we used the notation for a Kato log structure, but we can equivalently use DF log structures). A morphism $(S, M, \alpha) \rightarrow (T, N, \beta)$ of log spaces is a morphism $f: S \rightarrow T$ in **Esp**, together with a morphism $f^{-1}(N, \beta) \rightarrow (M, \alpha)$ of log structures over S (i.e. for the topos \mathfrak{X}_S).

Spaces of Kato charts. Suppose that P is a finitely generated monoid. Consider the sheaf of monoids on **Esp** that sends an object S into the monoid $\mathrm{Hom}(P, \mathcal{O}(S))$; this is representable by a monoid object $\mathbb{A}(P)$ in **Esp**, which is constructed as follows (see also [GMa, Section 4.4], from where, again, we also borrow the notation).

If $P = \mathbb{N}^r$ is a free monoid, then $\mathrm{Hom}(P, \mathcal{O}(S)) = \mathcal{O}(S)^r$, so $\mathbb{A}(P) = (\mathbb{A}^1)^r$. In general, by Rédei's Theorem [Réd14, Theorem 72] the monoid P is finitely presented, so it the coequalizer of two homomorphisms $\mathbb{N}^s \rightrightarrows \mathbb{N}^r$. Hence $\mathrm{Hom}(P, \mathcal{O}(S))$ is represented by the equalizer the two arrows of $\mathbb{A}(\mathbb{N}^r) \rightrightarrows \mathbb{A}(\mathbb{N}^s)$ induced by the homomorphisms above, which exists by hypothesis.

The definition of $\mathbb{A}(P)$ equips it with a natural log structure, induced by the universal homomorphism $P \rightarrow \mathcal{O}(\mathbb{A}(P))$. A morphism of log spaces $(S, M, \alpha) \rightarrow \mathbb{A}(P)$ corresponding to compatible homomorphisms $P \rightarrow \mathcal{O}(S)$ and $P \rightarrow M(S)$ is strict if and only if $P \rightarrow M(S)$ is a Kato chart for (M, α) on S .

The group of invertible elements in $\mathrm{Hom}(P, \mathcal{O}(S))$ is

$$\begin{aligned} \mathrm{Hom}(P, \mathcal{O}^\times(S)) &= \mathrm{Hom}(P^{\mathrm{gp}}, \mathcal{O}^\times(S)) \\ &= \mathrm{Hom}(P^{\mathrm{gp}}, \mathcal{O}(S)) \end{aligned}$$

and hence the group functor that sends S into $\mathrm{Hom}(P, \mathcal{O}^\times(S))$ is represented by $\mathbb{G}(P) := \mathbb{A}(P^{\mathrm{gp}})$. There is an obvious action of $\mathbb{G}(P)$ on $\mathbb{A}(P)$.

Stacks of DF charts. Consider the stack $\mathrm{Div} = [\mathcal{O}/\mathcal{O}^\times] = [\mathbb{A}^1/\mathbb{G}_m]$ over the site **Esp**, and a finitely generated monoid P . There is a stack $\mathcal{A}(P)$ over **Esp**, whose sections over an object S of **Esp** consist of symmetric monoidal functors $P \rightarrow \mathrm{Div}(S)$.

Proposition 2.18. *Suppose that P is fine and torsion-free. Then the stack $\mathcal{A}(P)$ is the quotient $[\mathbb{A}(P)/\mathbb{G}(P)]$.*

This is the analogue of [BV12, Proposition 3.25].

Proof. Denote by $\pi: \mathcal{O} \rightarrow \mathrm{Div}$ the projection. There is an obvious map $\pi_P: \mathbb{A}(P) \rightarrow \mathcal{A}(P)$ that sends a homomorphism $\phi: P \rightarrow \mathcal{O}(S)$ into the composite $\pi \circ \phi$. Since $\pi_P: \mathbb{A}(P) \rightarrow \mathcal{A}(P)$ is an epimorphism of stacks (that is, it admits local sections) by Lemma 2.13 (this is where we are using the torsion-free condition), we have that $\mathcal{A}(P)$ is the quotient of the groupoid

$$\mathbb{A}(P) \times_{\mathcal{A}(P)} \mathbb{A}(P) \rightrightarrows \mathbb{A}(P);$$

hence it enough to prove that the groupoid above is isomorphic to the groupoid

$$\mathbb{A}(P) \times \mathbb{G}(P) \rightrightarrows \mathbb{A}(P)$$

defined by the action of $\mathbb{G}(P)$.

Suppose that (ϕ, α) is an element of $(\mathbb{A}(P) \times \mathbb{G}(P))(S)$; in other words, $\phi: P \rightarrow \mathcal{O}(S)$ and $\alpha: P \rightarrow \mathcal{O}^\times(S)$ are homomorphisms of monoids. Then the product $\phi\alpha: P \rightarrow \mathcal{O}(S)$ is another section of $\mathbb{A}(P)(S)$, and α gives an isomorphism of $\pi_P \circ \phi$ with $\pi_P \circ (\phi\alpha)$; so we obtain an element $(\phi, \phi\alpha, \alpha) \in (\mathbb{A}(P) \times_{\mathcal{A}(P)} \mathbb{A}(P))(S)$. This gives a map from $\mathbb{A}(P) \times \mathbb{G}(P)$ to $\mathbb{A}(P) \times_{\mathcal{A}(P)} \mathbb{A}(P)$, which is easily checked to be an isomorphism. \square

A DF structure on a object \mathcal{S} of $\mathbf{Stack}_{\mathbf{Esp}}$ (the 2-category of stacks over the site \mathbf{Esp}) can be defined as a collection of DF structures on all pairs (S, f) of spaces $S \in \mathbf{Esp}$ with a map $f: S \rightarrow \mathcal{S}$, and compatible isomorphisms between the various pullbacks. The fact that

$$\mathrm{Hom}_{\mathbf{Stack}_{\mathbf{Esp}}}(S, \mathcal{A}(P)) = \mathrm{Hom}_{\mathbf{SymMon}}(P, \mathrm{Div}(S))$$

for P torsion-free, equips $\mathcal{A}(P)$ with a tautological DF structure, and makes it the stack of DF charts for log spaces in \mathbf{Esp} . In other words, DF charts on a log space (S, M, α) in \mathbf{Esp} from a fine torsion-free monoid P correspond exactly to strict morphisms $S \rightarrow \mathcal{A}(P)$.

2.5. Root stacks. We can extend the definition of root stacks given in [BV12] to the axiomatic setting. Let (S, A, L) be a log space in a category \mathbf{Esp} as in Assumption 2.17. Let $A \rightarrow B$ be a homomorphism of sheaves of monoids in \mathfrak{X}_S .

The following definition is an obvious extension of [BV12, Definition 4.16] to our situation; see this paper if more details are needed.

Definition 2.19. The *root stack* $\sqrt[n]{(S, A, L)}$ is the stack over \mathbf{Esp} defined as follows.

An object of $\sqrt[n]{(S, A, L)}$ over a space T is a triple (f, N, α) , where $f: T \rightarrow S$ is an arrow in \mathbf{Esp} , $N: f^{-1}B \rightarrow \mathrm{Div}_T$ is a symmetric monoidal functor with trivial kernel, and α is an isomorphism of log structures between the restriction of $(f^{-1}A, N|_{f^{-1}A})$ and $f^{-1}(A, L)$.

The arrows are defined in the obvious way.

Notice that if $A \rightarrow B \rightarrow C$ are maps of sheaves of monoids on S , we have obvious functors $\sqrt[n]{(S, A, L)} \rightarrow \sqrt[n]{(S, B, L)}$.

We will be mostly interested in two classes of examples.

Assume that A is saturated. If n is a positive number, consider the subsheaf $\frac{1}{n}A \subseteq A^{\mathrm{gp}} \otimes \mathbb{Q}$, with the natural embedding $A \subseteq \frac{1}{n}A$. (Of course we can think of $\frac{1}{n}A$ as A itself, with the map $A \rightarrow \frac{1}{n}A$ given by multiplication by n .) We denote the resulting stack by $\sqrt[n]{(S, A, L)} = \sqrt[n]{S}$. Notice that $\sqrt[n]{S}$ comes with a tautological log structure, whose sheaf of monoids is the pullback of $\frac{1}{n}A$ from S .

Here is a local description of $\sqrt[n]{S}$. Suppose that there is a chart $P \rightarrow A(S)$, where P is a fine saturated monoid. The embedding $P \subseteq \frac{1}{n}P$ gives compatible maps $\mathbb{A}(\frac{1}{n}P) \rightarrow \mathbb{A}(P)$ and $\mathbb{G}(\frac{1}{n}P) \rightarrow \mathbb{G}(P)$, which induce a morphism of stacks $\mathcal{A}(\frac{1}{n}P) \rightarrow \mathcal{A}(P)$. The log structure on $\sqrt[n]{S}$, with sheaf of monoids equal to the pullback of $\frac{1}{n}A$, gives a map $\sqrt[n]{S} \rightarrow \mathcal{A}(\frac{1}{n}P)$ with an isomorphism between the two composites $\sqrt[n]{S} \rightarrow \mathcal{A}(\frac{1}{n}P) \rightarrow \mathcal{A}(P)$ and $\sqrt[n]{S} \rightarrow S \rightarrow \mathcal{A}(P)$.

Proposition 2.20. *The diagram*

$$\begin{array}{ccc} \sqrt[n]{S} & \longrightarrow & \mathcal{A}(\frac{1}{n}P) \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{A}(P) \end{array}$$

is cartesian.

The proof is straightforward (cfr. [BV12, Proposition 4.13]).

In particular, if the chart $S \rightarrow \mathcal{A}(P)$ factors through a Kato chart $S \rightarrow \mathbb{A}(P)$, we have an isomorphism $\sqrt[n]{S} = [S \times_{\mathbb{A}(P)} \mathbb{A}(\frac{1}{n}P) / \mu_n(P)]$, where $\mu_n(P)$ is the group object $\mathbb{G}(C_n(P))$ associated with the cokernel $C_n(P)$ of the homomorphism $P^{\text{gp}} \rightarrow \frac{1}{n}P^{\text{gp}}$, and the action on $S \times_{\mathbb{A}(P)} \mathbb{A}(\frac{1}{n}P)$ is induced by the action on the second factor.

Given a sheaf of integral monoids A on S , consider the submonoid $A_{\mathbb{Q}} \rightarrow A^{\text{gp}} \otimes \mathbb{Q}$, defined as the subsheaf of sections a of $A^{\text{gp}} \otimes \mathbb{Q}$ with the property that locally on S there is a positive natural number q such that qs is in $A \subseteq A^{\text{gp}} \otimes \mathbb{Q}$.

Definition 2.21. The *infinite root stack* $\sqrt[\infty]{(S, A, L)} = \sqrt[\infty]{S}$ is ${}^A\sqrt{(S, A, L)}$.

We have the following alternate characterization. The embeddings $\frac{1}{n}A \subseteq A_{\mathbb{Q}}$ induce compatible functors $\sqrt[\infty]{S} \rightarrow \sqrt[n]{S}$, and therefore a morphism $\sqrt[\infty]{S} \rightarrow \varprojlim_n \sqrt[n]{S}$ (here the indexing system for the projective limit is the set of positive integers, ordered by divisibility). The following is a straightforward generalization of [TV, Proposition 3.5].

Proposition 2.22. *The morphism $\sqrt[\infty]{S} \rightarrow \varprojlim_n \sqrt[n]{S}$ above is an equivalence of stacks over \mathbf{Esp} .*

It should be remarked that $\sqrt[\infty]{S}$ is not always well behaved; when the category \mathbf{Esp} is too small, for example it does not admit countable projective limits, then $\sqrt[\infty]{S}$ may have too few objects (see for example Remark 2.24). In these cases one should probably define the infinite root stack to be the projective system given by the $\sqrt[n]{S}$.

2.6. Log algebraic, analytic and topological spaces. Let us specialize the general definitions to algebraic, analytic and topological spaces.

We will first briefly recall the notions of analytic and topological stacks, and the extension of the analytification functor. We refer the reader to [Noo05] for details, especially about the latter case.

As algebraic stacks over schemes on some base S are defined as categories fibered in groupoids over (Sch/S) that satisfy a gluing condition and are presented by a groupoid $R \rightrightarrows U$ with “nice” structure maps (typically étale or smooth), analytic and topological stacks are defined in the same way by switching schemes with the appropriate kind of object.

For analytic stacks we consider the site of analytic spaces with the classical topology, and consider stacks that are presented by groupoids $R \rightrightarrows U$ where the structure maps are holomorphic submersions. We will use the term “Deligne–Mumford” to indicate stacks that can be presented with a groupoid where the structure maps are étale.

For the topological case, a topological stack will be a stack on the site of topological spaces with the classical topology, and admitting a presentation by a groupoid $R \rightrightarrows U$ with structure maps that are “locally cartesian maps with Euclidean fibers” - the analogue in this context of smooth maps (see [Noo05]). We will say that a topological stack is “Deligne–Mumford” if it can be presented by a groupoid with étale structure maps (i.e. local homeomorphisms).

There is an analytification functor that produces an analytic stack from an algebraic stack (locally of finite type over \mathbb{C}), and an “underlying topological stack” functor that produces a topological stack from an analytic stack. They both extend the natural analytification functor on schemes of finite type over \mathbb{C} and “underlying topological space” functor on analytic spaces, respectively.

We will briefly sketch the construction of the analytification functor (the other case is analogous), and refer the reader to [Noo05, Theorem 20.1] for more details. We will apply the same process to the “Kato–Nakayama functor” in Section 4 in order to extend it to log algebraic stacks, and give a slightly more detailed proof (see Theorem 4.1).

Given an algebraic stack \mathcal{X} locally of finite type over \mathbb{C} , we want to produce an analytic stack $(\mathcal{X})_{\text{an}}$. Let us choose a presenting groupoid $R \rightrightarrows U$ for \mathcal{X} , and consider the induced groupoid $R_{\text{an}} \rightrightarrows U_{\text{an}}$. This is a groupoid in analytic spaces, whose structure maps are holomorphic submersions. Hence the quotient $[R_{\text{an}}/U_{\text{an}}]$ is an analytic stack, that we take to be the analytification $(\mathcal{X})_{\text{an}}$. One can check that the construction does not depend on the presenting groupoid (up to unique isomorphism), and that this extends to a functor from algebraic stacks to analytic stacks.

A more conceptual proof can be given along the lines of [CSST, Theorem 3.1], by constructing $(\mathcal{X})_{\text{an}}$ via the left Kan extension of $(-)_{\text{an}}$ along the Yoneda embedding. This gives for $(\mathcal{X})_{\text{an}}$ the “explicit” formula

$$(\mathcal{X})_{\text{an}} = \varinjlim_{\text{Spec } R \rightarrow \mathcal{X}} (\text{Spec } R)_{\text{an}}$$

where the colimit is a lax colimit in the 2-category of analytic stacks.

Let us go back to specializing our abstract setting about log structures. From now on log structures will be fine and saturated. In particular we can find local charts from monoids that are fine, saturated and sharp, hence torsion-free.

Remark 2.23. Log structures in the analytic context have already been considered in the literature, see for example [IKN05], and our notion coincides with the usual one. Log structures in a topological setting were considered, with a different spirit, in [Rog09]. We do not know what kind of relations there are between Rognes’s definition and ours, if any.

Algebraic. In the algebraic case we are taking \mathbf{Esp} to be the category of schemes (or more generally algebraic spaces) over a field k (or a more general base) with the étale topology, and \mathbb{A}_k^1 as monoid object (the operation is given by multiplication). The scheme $\mathbb{A}(P)$ is $\text{Spec } k[P]$, and the stack $\mathcal{A}(P)$ is the quotient $[\text{Spec } k[P]/\widehat{P}]$, where \widehat{P} is the diagonalizable group associated with P^{gp} (these are exactly the stacks of charts of [BV12]).

If X is a fine saturated log algebraic space, the root stack $\sqrt[n]{X}$ is an algebraic stack (Deligne–Mumford in good characteristic), and has local presentations where X has a Kato chart $X \rightarrow \operatorname{Spec} k[P]$, as the quotient $[X \times_{\operatorname{Spec} k[P]} \operatorname{Spec} k[\frac{1}{n}P]/\mu_n(P)]$, where $\mu_n(P)$ is the Cartier dual of the cokernel of $P^{\text{gp}} \rightarrow \frac{1}{n}P^{\text{gp}}$ and the action on $X \times_{\operatorname{Spec} k[P]} \operatorname{Spec} k[\frac{1}{n}P]$ is induced by the one on the second factor (cfr. [BV12, Proposition 4.13]).

In this case the infinite root stack $\sqrt[\infty]{X}$ is a pro-algebraic stack, and has been studied extensively in [TV].

Analytic. In this case, in the notation of (2.4) we are considering **Esp** to be the category of complex analytic spaces endowed with the classical topology, and as monoid object the analytic space \mathbb{C} (with multiplication). The realization $\mathbb{A}(P)$ of a monoid P is the analytic space $(\operatorname{Spec} \mathbb{C}[P])_{\text{an}}$, and the stack $\mathcal{A}(P)$ is the quotient $[(\operatorname{Spec} \mathbb{C}[P])_{\text{an}}/\widehat{P}_{\text{an}}]$.

We denote by \mathcal{A}_X the classical site of the analytic space X , and by $\operatorname{Div}_X^{\text{an}}$ the stack $[\mathcal{O}_X/\mathcal{O}_X^\times]$ over \mathcal{A}_X . In analogy to the algebraic case, it is not hard to prove that the stack $\operatorname{Div}_X^{\text{an}}$ parametrizes pairs (L, s) of a holomorphic line bundle with a global section, over the classical site \mathcal{A}_X .

In this case if X is a fine and saturated log analytic space, for every n the root stack $\sqrt[n]{X}$ is an analytic Deligne–Mumford stack, and locally where X has a Kato chart $X \rightarrow (\operatorname{Spec} \mathbb{C}[P])_{\text{an}}$ is isomorphic to $[X \times_{(\operatorname{Spec} \mathbb{C}[P])_{\text{an}}} (\operatorname{Spec} \mathbb{C}[\frac{1}{n}P])_{\text{an}}/(\mu_n(P))_{\text{an}}]$, where $\mu_n(P)$ is the Cartier dual of the cokernel of $P^{\text{gp}} \rightarrow \frac{1}{n}P^{\text{gp}}$ and the action on $X \times_{(\operatorname{Spec} \mathbb{C}[P])_{\text{an}}} (\operatorname{Spec} \mathbb{C}[\frac{1}{n}P])_{\text{an}}$ is induced by the one on the second factor.

Remark 2.24. There is a difference with the algebraic and the topological case regarding the infinite root stack $\sqrt[\infty]{X}$, that is worth a few words. In the analytic case, this stack has very few objects. This is due to the fact that analytic spaces are by definition locally of finite type, and thus they cannot have roots of every order of a non-zero non-constant holomorphic function.

Let us consider for example $X = \mathbb{C}$, with coordinate z , as a log analytic space with the log structure given by the origin, and its infinite root stack $\sqrt[\infty]{X}$. If $f: Y \rightarrow \mathbb{C}$ is a non-constant map from an analytic space Y that hits the origin, then the pullback to Y of the function z is a non-constant analytic function, and there cannot exist a sequence of analytic functions z_n on Y such that $z_n^n = f^*z$ for all n : the local ring $\mathcal{O}_{Y,y}$ in a point $y \in Y$ that maps to the origin is local Noetherian, and if there existed roots as above, then $f^*z \in \mathfrak{m}_y^n$ for every n , hence we would have $f^*z = 0$, as $\bigcap_n \mathfrak{m}_y^n = \{0\}$.

In this case $\sqrt[\infty]{X}$ is isomorphic to the disjoint union $\mathbb{C}^\times \bigsqcup \sqrt[\infty]{0}$, where the origin $0 \in \mathbb{C}$ is given the induced log structure. Because of this, it is best to see the infinite root stack as a pro-object instead than an actual stack, in this setting.

Topological. For the topological case we take as **Esp** the category of topological spaces with the classical topology, and as monoid object the topological space \mathbb{C} (with multiplication). The realization $\mathbb{A}(P)$ of a monoid P is the topological space $(\operatorname{Spec} \mathbb{C}[P])_{\text{top}}$, and the stack $\mathcal{A}(P)$ is the quotient $[(\operatorname{Spec} \mathbb{C}[P])_{\text{top}}/\widehat{P}_{\text{top}}]$. In the topological case we will write $\mathbb{C}(P)$ for $(\operatorname{Spec} \mathbb{C}[P])_{\text{top}}$.

In the topological world the substitute for the stack $\mathrm{Div}_X = [\mathcal{O}_X/\mathcal{O}_X^\times]$ over $X_{\mathrm{\acute{e}t}}$ is the quotient stack $\mathrm{Div}_T^{\mathrm{top}} = [\mathbb{C}/\mathbb{C}^\times]_T$ on the classical site \mathcal{C}_T of a topological space T , which turns out to parametrize continuous complex line bundles with a global section, in analogy with the algebraic and analytic cases.

If T is a fine saturated log topological space, for every n the n -th root stack $\sqrt[n]{T}$ is a topological Deligne–Mumford stack, and where T has a Kato chart $T \rightarrow \mathbb{C}(P)$ is isomorphic to the quotient stack $[T \times_{\mathbb{C}(P)} \mathbb{C}(\frac{1}{n}P)/(\mu_n(P))_{\mathrm{top}}]$, where $\mu_n(P)$ is the Cartier dual of the cokernel of $P^{\mathrm{gp}} \rightarrow \frac{1}{n}P^{\mathrm{gp}}$ and the action on $T \times_{\mathbb{C}(P)} \mathbb{C}(\frac{1}{n}P)$ is given by the one on the second factor.

Positive topological spaces. We remark that the formalism applies also to the case where **Esp** is the category of topological spaces, and the monoid object is $\mathbb{R}_{\geq 0}$ with multiplication (cfr. [GMa, Section 5.7]). In this case, the morphism α of a log structure takes value in the sheaf of real-valued positive functions of a space T , and the stack Div_S on an object $S \in \mathbf{Esp}$ is $[\mathbb{R}_{\geq 0}/\mathbb{R}_{> 0}]|_S$. This parametrizes real oriented line bundles.

The realization $\mathbb{A}(P)$ of the monoid P in **Esp** is the space $\mathbb{R}_{\geq 0}(P) = \mathrm{Hom}(P, \mathbb{R}_{\geq 0})$, and the stack $\mathcal{A}(P)$ is the quotient $[\mathbb{R}_{\geq 0}(P)/\mathbb{R}_{> 0}(P)]$. Note in particular that, by Proposition 2.18, for P fine and torsion-free there is an equivalence between maps $S \rightarrow [\mathbb{R}_{\geq 0}(P)/\mathbb{R}_{> 0}(P)]$ and symmetric monoidal functors $P \rightarrow [\mathbb{R}_{\geq 0}/\mathbb{R}_{> 0}]|_S$.

In this case root stacks are uninteresting, because raising to the n -th power $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a homeomorphism.

2.7. Comparison of root stacks. Let us compare the different notions of log structures that we just defined.

If X is an algebraic space, we consider log structures on the monoided topos $(X_{\mathrm{\acute{e}t}}, \mathcal{O}_X)$. The analytification X_{an} determines the monoided topos $(X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}})$ (here X_{an} has the classical topology), and there is a map of monoided topoi $(X_{\mathrm{an}}, \mathcal{O}_{X_{\mathrm{an}}}) \rightarrow (X_{\mathrm{\acute{e}t}}, \mathcal{O}_X)$, so that an algebraic log structure induces an analytic one by pulling back (see (2.3)). If X is a log algebraic space, the analytification X_{an} will be equipped with this induced log structure without further mention.

Analogously, if X is an analytic space there is a morphism of monoided topoi $(X_{\mathrm{top}}, \mathbb{C}_{X_{\mathrm{top}}}) \rightarrow (X, \mathcal{O}_X)$, so an analytic log structure induces a topological one, by pulling back along this morphism. Note that both of these operations preserve the existence of local charts, and the sheaf \overline{M} . Consequently, properties of the log structure such as being finitely generated, integral, saturated or coherent are also preserved.

We prove now that the three versions of the root stack construction are compatible with the “analytification” and “underlying topological space” functors.

Proposition 2.25. *Let X be a fine saturated log algebraic space locally of finite type over \mathbb{C} (resp. analytic space), and $n \in \mathbb{N}$ be a positive natural number.*

Then the analytic (resp. topological) stack $\sqrt[n]{X}_{\mathrm{an}}$ (resp. $\sqrt[n]{X}_{\mathrm{top}}$) associated with the n -th root stack of X is canonically isomorphic to the n -th root

stack $\sqrt[n]{X_{\text{an}}}$ (resp. $\sqrt[n]{X_{\text{top}}}$) of the associated log analytic space (resp. log topological space) of X .

In short, $\sqrt[n]{X_{\text{an}}} \cong \sqrt[n]{X_{\text{an}}}$ and $\sqrt[n]{X_{\text{top}}} \cong \sqrt[n]{X_{\text{top}}}$. This will be used to describe functorially the topological infinite root stack $\sqrt[\infty]{X_{\text{top}}} = \varprojlim_n \sqrt[n]{X_{\text{top}}}$ of a log analytic space (or log algebraic space) X .

Proof. The proof will be entirely analogous in the two cases, so we will carry it out only in the analytic case.

The general construction of $(\mathcal{X})_{\text{an}}$, if \mathcal{X} is any stack over schemes, is as a left Kan extension of $(-)_{\text{an}}$ along the Yoneda embedding (see [CSST, Section 3]). In other words we have the formula

$$(\mathcal{X})_{\text{an}} = \varinjlim_{\text{Spec } R \rightarrow \mathcal{X}} (\text{Spec } R)_{\text{an}}$$

where the colimit is a lax colimit in the 2-category of analytic stacks.

Now assume we are given a map $f: \text{Spec } R \rightarrow \sqrt[n]{X}$, and let us see how to produce a map $g: (\text{Spec } R)_{\text{an}} \rightarrow \sqrt[n]{X_{\text{an}}}$. By Yoneda and the functorial description of the two stacks, the map f corresponds to a morphism $\phi: \text{Spec } R \rightarrow X$ and a lifting of $\phi^{-1}L: \phi^{-1}A \rightarrow \text{Div}_R$ to $N: \frac{1}{n}\phi^{-1}A \rightarrow \text{Div}_R$. We need to produce a lifting $N_{\text{an}}: \frac{1}{n}\phi^{-1}A_{\text{an}} \rightarrow \text{Div}_R^{\text{an}}$ of $\phi^{-1}L_{\text{an}}: \phi^{-1}A_{\text{an}} \rightarrow \text{Div}_R^{\text{an}}$.

Let a be a section of $\frac{1}{n}\phi^{-1}A_{\text{an}}$ over some analytic open $U \subseteq (\text{Spec } R)_{\text{an}}$. Then we can find an étale $V \rightarrow \text{Spec } R$ with a section $\sigma: U \rightarrow V_{\text{an}}$ that is a homeomorphism onto the image, and a section b of $\frac{1}{n}\phi^{-1}A(V)$ that corresponds to a . The section b gives $N(b) = (L_b, s_b)$, a line bundle over V with a global section s . By analytifying, we get a complex line bundle $(L_b)_{\text{an}}$ with a global holomorphic section $(s_b)_{\text{an}}$. By restricting to U , this defines the image of a in $\text{Div}_R^{\text{an}}(U)$. This process extends in the obvious way to a symmetric monoidal functor of monoidal stacks over the analytic site $\mathcal{A}_{(\text{Spec } R)_{\text{an}}}$ that lifts $\phi^{-1}L_{\text{an}}$, i.e. a morphism $(\text{Spec } R)_{\text{an}} \rightarrow \sqrt[n]{X_{\text{an}}}$.

From this we get a morphism $\sqrt[n]{X_{\text{an}}} \rightarrow \sqrt[n]{X_{\text{an}}}$: every $\text{Spec } R \rightarrow X$ induces $(\text{Spec } R)_{\text{an}} \rightarrow \sqrt[n]{X_{\text{an}}}$ and by construction these are compatible with respect to commuting triangles. By the universal property of the colimit we obtain the desired morphism $\sqrt[n]{X_{\text{an}}} \rightarrow \sqrt[n]{X_{\text{an}}}$.

To check that it is an isomorphism we can do so locally on X , where there is a Kato chart $X \rightarrow \text{Spec } \mathbb{C}[P]$ for a fine torsion-free monoid P . In that case we have a quotient stack description of $\sqrt[n]{X}$ as

$$\sqrt[n]{X} = [X_n / \mu_n(P)]$$

where $X_n = X \times_{\text{Spec } \mathbb{C}[P]} \text{Spec } \mathbb{C}[\frac{1}{n}P]$ and $\mu_n(P)$ is the Cartier dual of the cokernel of $P^{\text{gp}} \rightarrow \frac{1}{n}P^{\text{gp}}$, that acts on X_n by acting on the second factor. From the construction of $(-)_{\text{an}}$ via presenting groupoids of [Noo05, Theorem 20.1], recalled in (2.6), it follows that $\sqrt[n]{X_{\text{an}}} = [(X_n)_{\text{an}} / \mu_n(P)_{\text{an}}]$.

From the analytic stack description of $\sqrt[n]{X_{\text{an}}}$ as a quotient in the presence of a global chart given in (2.6), we see that it coincides with the one just described. The map $\sqrt[n]{X_{\text{an}}} \rightarrow \sqrt[n]{X_{\text{an}}}$ in this local case is an isomorphism, and this concludes the proof. \square

3. THE KATO-NAKAYAMA SPACE AS A “ROOT STACK”

Let X be a fine saturated log analytic space. In this section we give a functorial description of the Kato-Nakayama space (see [KN99] or the Appendix of [CSST]) of X in the language of DF structures, and that bears a close similarity to the description of root stacks. It presents the Kato-Nakayama space as a sort of “transcendental” root stack.

As a byproduct of this alternative description we obtain a global construction of the canonical morphism $\Phi_X: X_{\log} \rightarrow (\sqrt[n]{X})_{\text{top}}$ of [CSST] ((3.4) and Proposition 4.6 below).

3.1. The case of a single divisor. Let us start with a motivating example.

Assume that X is a smooth analytic space with a log structure given by a single smooth divisor. In this case there is a global chart $X \rightarrow [\mathbb{C}/\mathbb{C}^\times]$ for the log structure, corresponding to the map $\mathbb{N} \rightarrow [\mathbb{C}/\mathbb{C}^\times](X)$ that sends 1 to $(\mathcal{O}_X(D), 1_D)$. Here we are considering \mathbb{C} and \mathbb{C}^\times as analytic spaces.

The various root stacks of X can be obtained as fibered products in the following manner (see (2.5)): if $\wedge n: [\mathbb{C}/\mathbb{C}^\times] \rightarrow [\mathbb{C}/\mathbb{C}^\times]$ is the map induced by “raising to the n -th power” on both the space and the group, we have a cartesian diagram

$$\begin{array}{ccc} \sqrt[n]{X} & \longrightarrow & [\mathbb{C}/\mathbb{C}^\times] \\ \downarrow & & \downarrow \wedge n \\ X & \longrightarrow & [\mathbb{C}/\mathbb{C}^\times]. \end{array}$$

The basic insight is that the Kato-Nakayama space can be obtained in a similar way as well. The idea for what follows is due to Kai Behrend.

Let us consider the “closed right half plane” $\mathbb{H} = \{(x, y) \in \mathbb{C} \mid x \geq 0\}$, with its operation given by $(x, y)(x', y') = (x \cdot x', y + y')$ that makes it a commutative monoid. There is an action of the group \mathbb{C}^+ of complex numbers with addition (we use this notation to distinguish it from the analytic space \mathbb{C}) on \mathbb{H} given by $(a + ib) \cdot (x, y) = (e^a \cdot x, b + y)$, and we will consider the quotient stack $[\mathbb{H}/\mathbb{C}^+]$ as a topological stack (note that the action is not via holomorphic functions).

The action of \mathbb{C}^+ on \mathbb{H} has two orbits: points with $x > 0$ have trivial stabilizer and the action is transitive among them, so they give a single open point of $[\mathbb{H}/\mathbb{C}^+]$. The other orbit is the y -axis, with stabilizer $\mathbb{R}^+ \subseteq \mathbb{C}^+$. So we can loosely write $[\mathbb{H}/\mathbb{C}^+] = * \cup \mathbb{B}\mathbb{R}^+$.

We have a morphism of stacks $\exp: [\mathbb{H}/\mathbb{C}^+] \rightarrow [\mathbb{C}/\mathbb{C}^\times]$ given by the exponential $\exp: \mathbb{C}^+ \rightarrow \mathbb{C}^\times$ at the level of groups, and by the \exp -equivariant map $\mathbb{H} \rightarrow \mathbb{C}$ that sends (x, y) to $x \cdot e^{iy}$ (which is itself a sort of exponential), at the level of spaces. This coincides with the universal cover of \mathbb{C}^\times if we restrict it to the complement of the y -axis, which in turn gets contracted to the origin in \mathbb{C} .

Now note that $[\mathbb{C}/\mathbb{C}^\times]$ also has two points, namely $[\mathbb{C}/\mathbb{C}^\times] = * \cup \mathbb{B}\mathbb{C}^\times$, and the morphism $[\mathbb{H}/\mathbb{C}^+] \rightarrow [\mathbb{C}/\mathbb{C}^\times]$ “maps” $*$ to $*$ and $\mathbb{B}\mathbb{R}^+ \rightarrow \mathbb{B}\mathbb{C}^\times$, via $\exp: \mathbb{R}^+ \rightarrow \mathbb{C}^\times$. This last homomorphism is injective with cokernel isomorphic to S^1 .

Because of this description, the morphism $[\mathbb{H}/\mathbb{C}^+] \rightarrow [\mathbb{C}/\mathbb{C}^\times]$ is an isomorphism over the open point and an S^1 -bundle over the closed point. Since the map $X \rightarrow [\mathbb{C}/\mathbb{C}^\times]$ sends $X \setminus D$ to the open point and D to the closed point, it is apparent that by pulling back we will find precisely the Kato-Nakayama space (i.e. the real oriented blow up, in this case), so that there should be (see (3.3) below for the proof) a cartesian diagram

$$\begin{array}{ccc} X_{\log} & \longrightarrow & [\mathbb{H}/\mathbb{C}^+] \\ \downarrow & & \downarrow \text{exp} \\ X & \longrightarrow & [\mathbb{C}/\mathbb{C}^\times]. \end{array}$$

Moreover note that $[\mathbb{H}/\mathbb{C}^+] \rightarrow [\mathbb{C}/\mathbb{C}^\times]$ factors as $[\mathbb{H}/\mathbb{C}^+] \rightarrow [\mathbb{C}/\mathbb{C}^\times] \xrightarrow{\wedge n} [\mathbb{C}/\mathbb{C}^\times]$ for every n by sending $(x, y) \in \mathbb{H}$ to $\sqrt[n]{x} \cdot e^{i\frac{y}{n}} \in \mathbb{C}$ and using $\exp(\frac{\cdot}{n}) : \mathbb{C}^+ \rightarrow \mathbb{C}^\times$ on the groups.

This will give a morphism $X_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$ for every n (here we are using Proposition 2.25), that all together will give a morphism $X_{\log} \rightarrow \varprojlim_n (\sqrt[n]{X})_{\text{top}} = \sqrt[\infty]{X}_{\text{top}}$ of topological stacks, in this special case.

3.2. The general case. Let us use the language of DF structures to generalize the above construction.

The log structure of X is given by a morphism $A \rightarrow \text{Div}_X^{\text{an}}$ of symmetric monoidal stacks on the analytic site \mathcal{A}_X . The n -th root stack $\sqrt[n]{X}$ parametrizes liftings of the log structure to the sheaf of formal fractions $\frac{1}{n}A$, i.e. diagrams

$$\begin{array}{ccc} A & \longrightarrow & \text{Div}_X^{\text{an}} \\ \downarrow & \nearrow & \\ \frac{1}{n}A & & \end{array}$$

(over some analytic space over X) or, alternatively, liftings

$$\begin{array}{ccc} A & \longrightarrow & \text{Div}_X^{\text{an}} \\ & \searrow & \uparrow \wedge n \\ & & \text{Div}_X^{\text{an}} \end{array}$$

where $\wedge n : \text{Div}_X^{\text{an}} \rightarrow \text{Div}_X^{\text{an}}$ sends (L, s) into $(L^{\otimes n}, s^{\otimes n})$.

There is a description of the Kato-Nakayama space in this spirit, that uses the symmetric monoidal stack $[\mathbb{H}/\mathbb{C}^+]$ introduced in the previous section (which turns out to “dominate” every such root morphism $\wedge n$, as we already explained above and will discuss in more detail in (3.4)).

Definition 3.1. Let us consider the stack $X_{\mathbb{H}}$ over the category of *topological spaces* over X_{top} that sends a space $\phi : T \rightarrow X_{\text{top}}$ to the groupoid of liftings

$$\begin{array}{ccc} \phi^{-1}A & \longrightarrow & [\mathbb{C}/\mathbb{C}^\times]_T \\ & \searrow & \uparrow \text{exp} \\ & & [\mathbb{H}/\mathbb{C}^+]_T \end{array}$$

where $\phi^{-1}A \rightarrow [\mathbb{H}/\mathbb{C}^+]_T$ is a symmetric monoidal functor. The arrows between the objects are given by the obvious natural transformations.

Here the map $\phi^{-1}A \rightarrow [\mathbb{C}/\mathbb{C}^\times]_T$ is the pullback to T of the topological DF structure on X_{top} induced by the given analytic DF structure on X .

The stack $X_{\mathbb{H}}$ parametrizes liftings of the \mathbb{C}^\times -torsors $(\phi^{-1}L)(a)$ to \mathbb{C}^+ -torsors along $\exp: \mathbb{C}^+ \rightarrow \mathbb{C}^\times$, equipped with a \mathbb{C}^+ -equivariant map to \mathbb{H} that covers the given \mathbb{C}^\times -equivariant map $(\phi^{-1}L)(a) \rightarrow \mathbb{C}$.

Theorem 3.2. *Let X be a fine saturated log analytic space. The stack $X_{\mathbb{H}}$ is represented by the Kato-Nakayama space X_{\log} , i.e. there is a canonical isomorphism of topological stacks $X_{\mathbb{H}} \cong X_{\log}$ over X_{top} .*

The starting point of the proof is the following functorial characterization of X_{\log} .

Theorem 3.3 ([IKN05, (1.2)]). *Consider the functor F_{\log} that sends a topological space $\phi: T \rightarrow X_{\text{top}}$ to the set of morphisms of sheaves of abelian groups $c: \phi^{-1}M^{\text{gp}} \rightarrow S_T^1$ such that $c(\phi^{-1}f) = f/|f|$ for $f \in \mathcal{O}_X^\times$, and that acts in the obvious way on the arrows.*

Then F_{\log} is represented by the Kato-Nakayama space X_{\log} .

Proof of Theorem 3.2. Let us describe concretely how the analytic log structure $\alpha: M \rightarrow \mathcal{O}_X$ on X induces a topological log structure on X_{top} . We take the composite $\beta: M \rightarrow \mathbb{C}_{X_{\text{top}}}$ of α and the natural map $\mathcal{O}_X \rightarrow \mathbb{C}_{X_{\text{top}}}$, and then form the amalgamated sum $M_{\text{top}} = M \oplus_{\beta^{-1}(\mathbb{C}_{X_{\text{top}}}^\times)} \mathbb{C}_{X_{\text{top}}}^\times$. The induced map $\alpha_{\text{top}}: M_{\text{top}} \rightarrow \mathbb{C}_{X_{\log}}$ gives the topological log structure. Note that we have an isomorphism between the characteristic shaves $\overline{M} \cong \overline{M_{\text{top}}}$, induced by $M \rightarrow M_{\text{top}}$.

We will rephrase the functorial interpretation of Theorem 3.3 in the language of DF structures. First note that since S^1 is a group, we have $\text{Hom}(\phi^{-1}M^{\text{gp}}, S_T^1) = \text{Hom}(\phi^{-1}M, S_T^1)$ and this is compatible with the condition on sections of \mathcal{O}_X^\times .

We claim that the set of homomorphisms

$$\left\{ c \in \text{Hom}(\phi^{-1}M, S_T^1) \mid c(\phi^{-1}f) = \frac{f}{|f|} \text{ for } f \in \mathcal{O}_X^\times \right\}$$

is the same as the set of morphisms of sheaves of monoids $d: \phi^{-1}M \rightarrow (\mathbb{R}_{\geq 0} \times S^1)_T$ such that $d(\phi^{-1}f) = (|f|, f/|f|)$ for $f \in \mathcal{O}_X^\times$ and the composite $\phi^{-1}M \rightarrow (\mathbb{R}_{\geq 0} \times S^1)_T \rightarrow (\mathbb{R}_{\geq 0})_T$ is $|\alpha_{\text{top}}|$ (where $|\cdot|$ denotes the usual euclidean absolute value on \mathbb{C}).

Given such a d , we can compose with the second projection $(\mathbb{R}_{\geq 0} \times S^1)_T \rightarrow S_T^1$ and obtain a $c \in \text{Hom}(\phi^{-1}M, S_T^1)$ satisfying the condition above. In the other direction, given $c: \phi^{-1}M \rightarrow S_T^1$, one can define the corresponding d via $d(m) = (|\alpha_{\text{top}}(m)|, c(m))$.

Now we claim that morphisms $d: \phi^{-1}M \rightarrow (\mathbb{R}_{\geq 0} \times S^1)_T$ as above correspond to symmetric monoidal functors

$$\overline{d}: \phi^{-1}\overline{M} \rightarrow [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]_T$$

that lift the DF structure $L_{\text{top}}: \phi^{-1}\overline{M} \rightarrow [\mathbb{C}/\mathbb{C}^\times]_T$ associated with α_{top} . Here the action of $\mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$ on $\mathbb{R}_{\geq 0} \times S^1$ is given by multiplication

on the two factors and $[\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]_T \rightarrow [\mathbb{C}/\mathbb{C}^\times]_T$ is induced by the \mathbb{C}^\times -equivariant function $\mathbb{R}_{\geq 0} \times S^1 \rightarrow \mathbb{C}$ sending (r, a) to $r \cdot a \in \mathbb{C}$.

First observe that, by construction of the sheaf M_{top} and the log structure α_{top} , there is a bijection between maps $d: \phi^{-1}M \rightarrow (\mathbb{R}_{\geq 0} \times S^1)_T$ such that $d(\phi^{-1}f) = (|f|, f/|f|)$ for every $f \in \mathcal{O}_X^\times$ and maps $\tilde{d}: \phi^{-1}M_{\text{top}} \rightarrow (\mathbb{R}_{\geq 0} \times S^1)_T$ such that $\tilde{d}(f) = (|f|, f/|f|)$ for every $f \in \mathbb{C}_T^\times$.

Now we follow the proof of Proposition 2.6. Note that the group \mathbb{C}_T^\times acts on both $\phi^{-1}M_{\text{top}}$ and $(\mathbb{R}_{\geq 0} \times S^1)_T$, and moreover the action on $\phi^{-1}M_{\text{top}}$ is free, with quotient $\phi^{-1}\overline{M}$. By taking the stacky quotient of \tilde{d} by this action we get a symmetric monoidal functor

$$\overline{d}: \phi^{-1}\overline{M} \rightarrow [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]_T.$$

Observe also that the composite $\phi^{-1}\overline{M} \xrightarrow{\overline{d}} [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]_T \rightarrow [\mathbb{C}/\mathbb{C}^\times]_T$ is naturally identified with $\phi^{-1}L_{\text{top}}$, where L_{top} is the DF structure associated with α_{top} .

The inverse construction is obtained by taking the base change of such a \overline{d} along the projection $(\mathbb{R}_{\geq 0} \times S^1)_T \rightarrow [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]_T$, which is a \mathbb{C}_T^\times -torsor.

Finally we note that there is an isomorphism of symmetric monoidal stacks

$$[\mathbb{H}/\mathbb{C}^+] \cong [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times],$$

where the action on the left is the same as in (3.1). The subgroup $j: \mathbb{Z} \subseteq \mathbb{C}^+$ given by $k \mapsto 2k\pi i$ acts without stabilizers on \mathbb{H} , and the quotient is $\mathbb{H}/\mathbb{Z} = \mathbb{R}_{\geq 0} \times S^1$. Moreover the cokernel of j is \mathbb{C}^\times (and the map is given by the exponential), and therefore

$$[\mathbb{H}/\mathbb{C}^+] \cong [(\mathbb{H}/\mathbb{Z})/(\mathbb{C}^+/\mathbb{Z})] \cong [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]$$

as symmetric monoidal stacks.

This also gives an isomorphism of symmetric monoidal stacks $[\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]_T \cong [\mathbb{H}/\mathbb{C}^+]_T$ over the site \mathcal{C}_T , which is compatible with the natural maps to $[\mathbb{C}/\mathbb{C}^\times]_T$. This shows that the functorial description of Theorem 3.3 coincides with the one of the stack $X_{\mathbb{H}}$ that we introduced above, and concludes the proof. \square

Remark 3.4. With the same reasoning as in the proof, we also have

$$[\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times] \cong [\mathbb{R}_{\geq 0}/\mathbb{R}_{>0}]$$

(by writing $\mathbb{C}^\times = \mathbb{R}_{>0} \times S^1$ and cancelling the S^1 factor).

3.3. Charts. In the spirit of the functorial interpretation of Theorem 3.2, we can obtain “charts” for the Kato-Nakayama space X_{\log} out of charts for the log structure of X .

Specifically, when X has a DF chart $X \rightarrow [(\text{Spec } \mathbb{C}[P])_{\text{an}}/\widehat{P}_{\text{an}}]$ with P fine and torsion-free, the cartesian diagram described in (3.1) can be replaced by the more general

$$(1) \quad \begin{array}{ccc} X_{\log} & \longrightarrow & [\mathbb{H}(P)/\mathbb{C}^+(P)] \\ \downarrow & & \downarrow \\ X_{\text{top}} & \longrightarrow & [\mathbb{C}(P)/\mathbb{C}^\times(P)] \end{array}$$

where $\mathbb{H}(P) = \text{Hom}(P, \mathbb{H})$ and $\mathbb{C}^+(P) = \text{Hom}(P, \mathbb{C}^+)$ have their natural topologies and monoid or group structures. The vertical map is given by composition with the exponential maps $\mathbb{H} \rightarrow \mathbb{C}$ and $\mathbb{C}^+ \rightarrow \mathbb{C}^\times$ that were discussed in (3.1), and for $P = \mathbb{N}$ the diagram reduces to the one showing up at the end of the discussion.

Remark 3.5. For every finitely generated monoid P there is an isomorphism

$$[\mathbb{H}(P)/\mathbb{C}^+(P)] \cong [(\mathbb{R}_{\geq 0} \times S^1)(P)/\mathbb{C}^\times(P)]$$

induced by the projection $\mathbb{H}(P) \rightarrow (\mathbb{R}_{\geq 0} \times S^1)(P)$ and the exponential $\mathbb{C}^+(P) \rightarrow \mathbb{C}^\times(P)$, as in the proof of Theorem 3.2. In an analogous way we also have an isomorphism

$$[(\mathbb{R}_{\geq 0} \times S^1)(P)/\mathbb{C}^\times(P)] \cong [\mathbb{R}_{\geq 0}(P)/\mathbb{R}_{> 0}(P)].$$

We can use any one of these models to describe charts for X_{\log} , and we will switch back and forth without further mention.

Proposition 3.6. *Let X be a fine saturated log analytic space equipped with a DF chart $X \rightarrow [(\text{Spec } \mathbb{C}[P])_{\text{an}}/\widehat{P}_{\text{an}}]$, with P fine and torsion-free. Then there is a natural diagram (1) as above, and it is cartesian.*

Proof. The point is to show that $[\mathbb{H}(P)/\mathbb{C}^+(P)] \rightarrow [\mathbb{C}(P)/\mathbb{C}^\times(P)]$ is the stack of charts for the objects parametrized by the Kato-Nakayama space, as in Theorem 3.2.

Given the isomorphisms

$$[\mathbb{H}/\mathbb{C}^+] \cong [\mathbb{R}_{\geq 0}/\mathbb{R}_{> 0}]$$

and

$$[\mathbb{H}(P)/\mathbb{C}^+(P)] \cong [\mathbb{R}_{\geq 0}(P)/\mathbb{R}_{> 0}(P)],$$

this follows from the discussion of stacks of DF charts in (2.4) in the case of “positive topological spaces”, and by arguing that for a monoid P and a topological space T , a morphism of symmetric monoidal categories $P \rightarrow [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times](T)$ can be sheafified to a morphism of symmetric monoidal stacks $A \rightarrow [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]_T$ with trivial kernel (see Propositions 2.4 and 2.10 of [BV12]), compatibly with the sheafification of the induced $P \rightarrow [\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times](T) \rightarrow [\mathbb{C}/\mathbb{C}^\times](T)$. \square

These kinds of charts are related to the usual ones for X_{\log} given by the topological space $\text{Hom}(P, \mathbb{R}_{\geq 0} \times S^1)$ (see for example [KN99, Section 1]) in the same way as DF charts are related to Kato charts for log spaces.

In fact for every fine monoid P the natural diagram

$$\begin{array}{ccc} (\mathbb{R}_{\geq 0} \times S^1)(P) & \longrightarrow & [(\mathbb{R}_{\geq 0} \times S^1)(P)/\mathbb{C}^\times(P)] \\ \downarrow & & \downarrow \\ \mathbb{C}(P) & \longrightarrow & [\mathbb{C}(P)/\mathbb{C}^\times(P)] \end{array}$$

is cartesian.

Note however that by using the presentation $[\mathbb{H}(P)/\mathbb{C}^+(P)]$ to compute the fibered product above, we would “spontaneously” end up with the diagram

$$\begin{array}{ccc} [\mathbb{H}(P)/\mathbb{Z}(P)] & \longrightarrow & [\mathbb{H}(P)/\mathbb{C}^+(P)] \\ \downarrow & & \downarrow \\ \mathbb{C}(P) & \longrightarrow & [\mathbb{C}(P)/\mathbb{C}^\times(P)] \end{array}$$

(the group $\mathbb{Z}(P)$ is the kernel of the surjective morphism $\exp(P): \mathbb{C}^+(P) \rightarrow \mathbb{C}^\times(P)$).

Of course there is an isomorphism $(\mathbb{R}_{\geq 0} \times S^1)(P) \cong [\mathbb{H}(P)/\mathbb{Z}(P)]$, but this gives some insight into the fact that the space $\mathbb{H}(P)$, that is used by Ogus in [Ogu03] (see in particular Section 3.1), is like an “atlas” for the Kato-Nakayama space in this language.

We can see an analogy with root stacks by looking at the presentations $\sqrt[n]{X} \cong [X_n/\mu_n(P)]$ given by a Kato chart $X \rightarrow \operatorname{Spec} \mathbb{C}[P]$. Here the atlas is $X_n = X \times_{\operatorname{Spec} \mathbb{C}[P]} \operatorname{Spec} \mathbb{C}[\frac{1}{n}P]$, and the group $\mu_n(P)$, which is the kernel of $\mathbb{C}^\times(P) \rightarrow \mathbb{C}^\times(P)$ induced by $z \mapsto z^n$, is the analogue of the group $\mathbb{Z}(P)$ (kernel of the exponential) above.

Remark 3.7 (Differentiable structure). The space X_{\log} has more structure than just that of a topological space, as may be apparent by staring at the charts we just described. The space \mathbb{H} has a smooth (even real analytic) structure (with a boundary), that is respected by the action of \mathbb{C}^+ .

In fact, as proven in [GMa, Section 6.8], the Kato-Nakayama space is naturally a *differentiable space* [Gil], and on top of that it carries a sort of “log structure” of its own. Precisely, it has a *positive log differentiable structure* [GMa, Section 6.1], meaning a log structure on the monoidal topos $(X_{\log}, \mathcal{R}_{X_{\log}}^{\geq 0})$, where for a differentiable space Y the sheaf $\mathcal{R}_Y^{\geq 0}$ on Y is the sheaf of functions of differentiable spaces to $\mathbb{R}_{\geq 0}$.

As mentioned in (2.6) (in the slightly different context of “positive topological spaces”), the stack of DF charts for this category of log structures would be $[\mathbb{R}_{\geq 0}(P)/\mathbb{R}_{> 0}(P)]$, and in fact the charts for X_{\log} described in above are of this form. The resulting structure coincides with the one of [GMa, Section 6.8], since the maps $X_{\log} \rightarrow [\mathbb{H}(P)/\mathbb{C}^+(P)]$ factor through the “Kato chart” given by $X_{\log} \rightarrow (\mathbb{R}_{\geq 0} \times S^1)(P) \rightarrow \mathbb{R}_{\geq 0}(P)$, in presence of a Kato chart $X \rightarrow (\operatorname{Spec} \mathbb{C}[P])_{\text{an}}$ for \bar{X} .

3.4. The map to the infinite root stack. Let us show how the functorial interpretation of Theorem 3.2 gives a globally defined morphism of topological stacks $X_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$ for every n , and these assemble into a morphism of pro-topological stacks $X_{\log} \rightarrow \sqrt[\infty]{X}_{\text{top}}$. We will check later (see Proposition 4.6) that this morphism coincides with the one constructed in [CSST, Proposition 4.1].

The point here is that the description of X_{\log} as a root stack allows us to canonically extract n -th roots, as follows: for a topological space $\phi: T \rightarrow X_{\text{top}}$ let us define

$$\Phi_n(T): X_{\log}(T) \rightarrow \sqrt[n]{X}_{\text{top}}(T)$$

by sending a morphism of symmetric monoidal categories $\phi^{-1}A \rightarrow [\mathbb{H}/\mathbb{C}^+]_T$ to the composite with the map $f_n: [\mathbb{H}/\mathbb{C}^+]_T \rightarrow [\mathbb{C}/\mathbb{C}^\times]_T$, induced by $\mathbb{H} \rightarrow \mathbb{C}$ that sends (x, y) to $\sqrt[n]{x} \cdot e^{iy/n}$ and $\mathbb{C}^+ \rightarrow \mathbb{C}^\times$ that sends z to $e^{z/n}$.

This gives an object of $\sqrt[n]{X}_{\text{top}}$. In fact we have a commutative diagram

$$\begin{array}{ccc} [\mathbb{H}/\mathbb{C}^+]_T & \xrightarrow{f_n} & [\mathbb{C}/\mathbb{C}^\times]_T \\ & \searrow \text{exp} & \downarrow \wedge n \\ & & [\mathbb{C}/\mathbb{C}^\times]_T \end{array}$$

that shows that $\phi^{-1}A \rightarrow [\mathbb{H}/\mathbb{C}^+]_T \xrightarrow{f_n} [\mathbb{C}/\mathbb{C}^\times]_T$ lifts the symmetric monoidal functor $\phi^{-1}L: \phi^{-1}A \rightarrow [\mathbb{C}/\mathbb{C}^\times]_T$ along the n -th power map $\wedge n: [\mathbb{C}/\mathbb{C}^\times]_T \rightarrow [\mathbb{C}/\mathbb{C}^\times]_T$. Here we are using the description of the functor of points of $\sqrt[n]{X}_{\text{top}}$ given by Proposition 2.25.

The resulting morphisms are compatible with respect to the projections $\sqrt[m]{X}_{\text{top}} \rightarrow \sqrt[n]{X}_{\text{top}}$ where $n \mid m$, and they give a morphism of pro-objects $X_{\log} \rightarrow \sqrt[\infty]{X}_{\text{top}}$.

Remark 3.8. As in the previous discussions, we can exchange $[\mathbb{H}/\mathbb{C}^+]$ with $[\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]$. Note however that the above maps cannot be defined as equivariant maps $\mathbb{R}_{\geq 0} \times S^1 \rightarrow \mathbb{C}$ (there is no section of the maps $z^n: S^1 \rightarrow S^1$ or $\mathbb{C}^\times \rightarrow \mathbb{C}^\times$).

We show in Proposition 4.6 below that the morphism $X_{\log} \rightarrow \sqrt[\infty]{X}_{\text{top}}$ that we just constructed coincides with the one of [CSST, Proposition 4.1].

Remark 3.9 (Real roots). The point of the above construction is to make use of the morphism $\phi_{\frac{1}{n}}: [\mathbb{H}/\mathbb{C}^+] \rightarrow [\mathbb{H}/\mathbb{C}^+]$ induced by the maps $\mathbb{H} \rightarrow \mathbb{H}$ acting as $(x, y) \mapsto (\sqrt[n]{x}, y/n)$ and $\mathbb{C}^+ \rightarrow \mathbb{C}^+$ given by $z \mapsto z/n$. This corresponds to extracting n -th roots, in that the diagram

$$\begin{array}{ccc} [\mathbb{H}/\mathbb{C}^+] & \xleftarrow{\phi_{\frac{1}{n}}} & [\mathbb{H}/\mathbb{C}^+] \\ \text{exp} \downarrow & & \downarrow \text{exp} \\ [\mathbb{C}/\mathbb{C}^\times] & \xrightarrow{\wedge n} & [\mathbb{C}/\mathbb{C}^\times] \end{array}$$

commutes.

More generally one can consider $\phi_r: [\mathbb{H}/\mathbb{C}^+] \rightarrow [\mathbb{H}/\mathbb{C}^+]$ for any $r \in \mathbb{R}_{>0}$, given in the same way by the maps $\mathbb{H} \rightarrow \mathbb{H}$, defined as $(x, y) \mapsto (x^r, ry)$, and $\mathbb{C}^+ \rightarrow \mathbb{C}^+$ given by $z \mapsto rz$.

Using these morphisms one can show that a lifting parameterized by the Kato-Nakayama space

$$\begin{array}{ccc} \phi^{-1}A & \xrightarrow{L} & [\mathbb{C}/\mathbb{C}^\times]_T \\ & \searrow L_{\mathbb{H}} & \uparrow \\ & & [\mathbb{H}/\mathbb{C}^+]_T \end{array}$$

will induce a 2-commutative diagram

$$\begin{array}{ccc} \phi^{-1}A & \xrightarrow{L} & [\mathbb{C}/\mathbb{C}^\times]_T \\ \downarrow & \nearrow L_{\mathbb{R}} & \\ \phi^{-1}A_{\mathbb{R}_{\geq 0}} & & \end{array}$$

that can be seen as a “real root” of the log structure, by setting

$$L_{\mathbb{R}}\left(\sum_i r_i \cdot a_i\right) = \exp(\phi_{r_1}(L_{\mathbb{H}}(a_1))) \otimes \cdots \otimes \exp(\phi_{r_k}(L_{\mathbb{H}}(a_k))).$$

where $r_i \in \mathbb{R}_{\geq 0}$ and a_i are sections of $\phi^{-1}A$.

It is not clear whether these two kinds of lifting can be identified completely, especially without imposing any “continuity” conditions on the second type of diagrams.

4. THE KATO-NAKAYAMA “SPACE” OF A LOG ALGEBRAIC STACK

In this final section we observe that the Kato-Nakayama construction applies also to log algebraic stacks that are locally of finite type over the complex numbers (and to log complex analytic stacks) and produces a topological stack, and we relate this to the “charts” for the Kato-Nakayama space described in (3.3). We also check that the morphism $X_{\log} \rightarrow \sqrt[\infty]{X}_{\text{top}}$ that was described in (3.4) coincides with the one of [CSST, Proposition 4.1].

Denote by \mathbf{Logst} the 2-category of locally of finite type fine log algebraic stacks over \mathbb{C} , and by \mathbf{Topst} the 2-category of topological stacks.

Theorem 4.1. *There is a morphism of 2-categories $(-)_\log: \mathbf{Logst} \rightarrow \mathbf{Topst}$ that preserves colimits, and extends the usual Kato-Nakayama functor on log algebraic spaces. This functor is uniquely determined (in the 2-categorical sense) by these properties.*

Remark 4.2. The preceding theorem is valid also if we replace the 2-category \mathbf{Logst} by the 2-category of fine log analytic stacks. Let us note that the analytification LOG_{an} of Olsson’s stack LOG (see [Ols03]) is the stack that parameterizes fine log structures on analytic spaces, that we denote temporarily by $LOG_{\mathbb{C}}$.

In fact, the discussion in [Ols03, Section 5] describes the stack LOG as the colimit in the category of stacks of the diagram, indexed by finitely generated integral monoids, of the toric stacks $[\text{Spec } \mathbb{C}[P]/\widehat{P}]$, with the natural maps between them. Since the local toric models are “the same”, that discussion applies also to the analytic stack $LOG_{\mathbb{C}}$ that parametrizes fine log structures on analytic spaces, which is then the colimit, indexed by the same category, of the stacks $[(\text{Spec } \mathbb{C}[P])_{\text{an}}/\widehat{P}_{\text{an}}]$. Finally the analytification functor preserves colimits, and there is a natural isomorphism $[\text{Spec } \mathbb{C}[P]/\widehat{P}]_{\text{an}} \cong [(\text{Spec } \mathbb{C}[P])_{\text{an}}/\widehat{P}_{\text{an}}]$ for any fine monoid P , so we obtain an induced canonical isomorphism $LOG_{\text{an}} \cong LOG_{\mathbb{C}}$.

The functor $(-)_\log$ can be applied to the analytic stack LOG_{an} to obtain a “universal” Kato-Nakayama space, in the sense that for every fine log

analytic space (or stack) X there is a cartesian square of topological stacks

$$\begin{array}{ccc} X_{\log} & \longrightarrow & LOG_{\log} \\ \downarrow & & \downarrow \\ X_{\text{top}} & \longrightarrow & LOG_{\text{top}}. \end{array}$$

Remark 4.3. As it happens for the Kato-Nakayama space (see Remark 3.7), for every log algebraic (or analytic) stack \mathcal{X} the topological stack \mathcal{X}_{\log} should have a structure of a “(positive log) differentiable stack” over the real numbers (the terminology is a bit awkward, since “differentiable stack” already has a meaning in the smooth differentiable world).

Proof of Theorem 4.1. We mimic the proof of [Noo05, Theorem 20.1].

Assume that $R \rightrightarrows U$ is a presentation of a log algebraic stack \mathcal{X} . Then both U and R have induced log structures, and the structure maps of the groupoid presentation are strict. Since the Kato-Nakayama functor preserves finite limits, we see that the resulting $R_{\log} \rightrightarrows U_{\log}$ is a groupoid in topological spaces.

Since the structure maps of $R \rightrightarrows U$ are smooth and strict, the structure maps of $R_{\log} \rightrightarrows U_{\log}$ are “locally cartesian maps with euclidean fibers” in Noohi’s terminology (see [Noo05]). We define \mathcal{X}_{\log} to be the quotient stack $[U_{\log}/R_{\log}]$. We sketch an argument to justify that this is independent of the groupoid presentation and extends to a functor, leaving most of the 2-categorical details to the reader.

Given another presenting groupoid $R' \rightrightarrows U'$ of \mathcal{X} , we can find a third one $R'' \rightrightarrows U''$ that has a map to both of these, inducing isomorphisms between the associated stacks. We will check that the induced morphisms $[U''_{\log}/R''_{\log}] \rightarrow [U'_{\log}/R'_{\log}]$ and $[U''_{\log}/R''_{\log}] \rightarrow [U_{\log}/R_{\log}]$ are isomorphisms. In this way we get an isomorphism $[U'_{\log}/R'_{\log}] \rightarrow [U_{\log}/R_{\log}]$ that depends on the choice of the third groupoid, but is unique up to a unique isomorphism. This defines the functor on objects.

Let us check that a map of groupoids $(R \rightrightarrows U) \rightarrow (R' \rightrightarrows U')$ in algebraic spaces that induces an isomorphism of quotient stacks gives an isomorphism $[U_{\log}/R_{\log}] \cong [U'_{\log}/R'_{\log}]$. We use the following fact (see [Sta16, Tag 046R]): a morphism of groupoids as above in an arbitrary site induces an isomorphism between the quotient stacks if and only if

- (i) the composite $t \circ \pi_1 : R' \times_{U'} U \rightarrow U'$ locally admits sections, and
- (ii) the natural map $R \rightarrow (U \times U) \times_{U' \times U'} R'$ is an isomorphism.

In our situation, since $(R \rightrightarrows U) \rightarrow (R' \rightrightarrows U')$ induces an isomorphism on the quotient stacks, we infer that (i) and (ii) hold. Using the fact that all maps are strict and by applying the functor $(-)_{\log}$ to all diagrams, we can conclude that (i) and (ii) also hold for the map of groupoids $(R_{\log} \rightrightarrows U_{\log}) \rightarrow (R'_{\log} \rightrightarrows U'_{\log})$ in topological spaces. Thus, the induced $[U_{\log}/R_{\log}] \rightarrow [U'_{\log}/R'_{\log}]$ is an isomorphism.

On 1-arrows, given a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ we can find presenting groupoids of \mathcal{X} and \mathcal{Y} and a map between those, that induces f . We use the above construction to obtain a morphism $f_{\log} : \mathcal{X}_{\log} \rightarrow \mathcal{Y}_{\log}$, and this again turns out to be unique up to a unique isomorphism.

The effect on natural transformation is uniquely determined by the above. \square

Note that by construction for any log algebraic (or analytic) stack \mathcal{X} there is a projection $\mathcal{X}_{\log} \rightarrow \mathcal{X}_{\text{top}}$.

Remark 4.4. The proof is a bit curt, but on the other hand gives a clear picture of what the functor is doing practically.

A more detailed (and general) proof could be given along the lines of the one of Theorem 3.1 (and the discussion that follows) of [CSST], using more sophisticated machinery.

Example 4.5. The Kato-Nakayama space of $[\mathbb{A}^1/\mathbb{G}_m]$ is the topological stack $[\mathbb{R}_{\geq 0} \times S^1/\mathbb{C}^\times]$, for the usual action. More generally, for a fine saturated monoid P the Kato-Nakayama space of the quotient $[(\text{Spec } \mathbb{C}[P])_{\text{an}}/\widehat{P}_{\text{an}}]$ is the stack $[\mathbb{H}(P)/\mathbb{C}^+(P)]$ that appears in the discussion of (3.3).

Let us show that the morphism constructed in (3.4) coincides with the morphism Φ_X of [CSST, Section 4].

Proposition 4.6. *The morphism $X_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$ of (3.4) coincides with the morphism Φ_X constructed in [CSST, Proposition 4.1].*

Proof. First, in light of the construction of Theorem 4.1 we can reinterpret Proposition 4.4 of [CSST] as follows: if X is a fine saturated log algebraic space locally of finite type over \mathbb{C} (or a fine saturated log analytic space), then for every positive integer n the canonical morphism $(\sqrt[n]{X})_{\log} \rightarrow X_{\log}$ is an isomorphism.

Indeed, it is proven there that, locally where X has a Kato chart, and thus the root stack has a presentation $\sqrt[n]{X} = [X_n/\mu_n(P)]$, the map $(X_n)_{\log} \rightarrow X_{\log}$ is a $\mu_n(P)_{\text{an}}$ -torsor. On the other hand the stack $(\sqrt[n]{X})_{\log}$ is identified with the quotient stack $[(X_n)_{\log}/\mu_n(P)_{\text{an}}]$ (note $\mu_n(P)_{\log} = \mu_n(P)_{\text{an}}$ since the log structure is trivial), and hence turns out to be isomorphic to X_{\log} via the natural map.

Note that since for every n there is a projection $(\sqrt[n]{X})_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$, this produces a canonical morphism $X_{\log} \cong (\sqrt[n]{X})_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$, that manifestly coincides with the Φ_n constructed in [CSST, Section 4].

Now let us check that it also agrees with the natural transformation described in (3.4).

The point is the following commutative diagram:

$$\begin{array}{ccc}
 (\sqrt[n]{X})_{\log} & \xrightarrow{\quad} & [\mathbb{H}(\frac{1}{n}P)/\mathbb{C}^+(\frac{1}{n}P)] \\
 \cong \swarrow & \downarrow & \nearrow \\
 X_{\log} & \xrightarrow{\quad} & [\mathbb{H}(P)/\mathbb{C}^+(P)] \\
 \downarrow & \searrow & \downarrow \phi_{\frac{1}{n}}(P) \\
 & \sqrt[n]{X}_{\text{top}} & \xrightarrow{\quad} [\mathbb{C}(\frac{1}{n}P)/\mathbb{C}^\times(\frac{1}{n}P)] \\
 \downarrow & \swarrow & \downarrow \\
 X_{\text{top}} & \xrightarrow{\quad} & [\mathbb{C}(P)/\mathbb{C}^\times(P)]
 \end{array}$$

where $\phi_{\frac{1}{n}}(P)$ is defined as the analogous maps of Remark 3.9.

The morphism described in (3.4) is determined (using the functorial interpretation of X_{\log}) by the composite

$$X_{\log} \rightarrow [\mathbb{H}(P)/\mathbb{C}^+(P)] \xrightarrow{\phi_{\frac{1}{n}}(P)} [\mathbb{H}(\frac{1}{n}P)/\mathbb{C}^+(\frac{1}{n}P)] \rightarrow [\mathbb{C}(\frac{1}{n}P)/\mathbb{C}^\times(\frac{1}{n}P)]$$

via the universal property of the fibered product in the bottom horizontal square, whereas the one of [CSST, Proposition 4.1], as per the previous discussion, is determined by the composite

$$X_{\log} \rightarrow (\sqrt[n]{X})_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}.$$

The fact that the diagram is commutative implies that the two maps $X_{\log} \rightarrow \sqrt[n]{X}_{\text{top}}$ coincide up to equivalence. \square

Remark 4.7. To conclude, let us point out that if we equip every Kato-Nakayama space (or stack) X_{\log} with its sheaf of rings \mathcal{O}_X^{\log} defined in [KN99], then the isomorphism $X_{\log} \cong (\sqrt[n]{X})_{\log}$ is **not** an isomorphism of ringed topological stacks. This will be further explored in upcoming work, related to question (2) mentioned in the introduction.

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